$$\beta_1 = \tfrac{76}{144}, \qquad \beta_2 = \tfrac{102}{144}, \qquad \beta_3 = \tfrac{114}{144},$$

and by (21), (22) and (13),

$$h_1 dx = 6 \div \surd(38n), \quad h_2 dy = 6 \div \surd(51n), \quad h_3 dz = 6 \div \surd(57n),$$
$$h_1^2 x^2 = 36i^2 \div 38n, \quad h_2^2 y^2 = 36j^2 \div 51n, \quad h_3^2 z^2 = 36k^2 \div 57n.$$

Substituting these in (30) we find

$$\log w = \bar{1}.06711 - \tfrac{3}{2}\log n - \frac{1}{n}(.41143 i^2 + .30656 j^2 + .27429 k^2),$$

from which the values of w may be computed by assigning suitable values to i, j, k, which in this example are any whole numbers, either + or — It would be interesting to have the polynomial raised algebraically to a power high enough to show the agreement between the true coefficients in the expansion, and their approximate values as given by the above formula; but this is impracticable, owing to the tedious length of the work required in forming the algebraic expansion.

The exponential function (30), regarded as the law of probability of error in space, has been reached by previous writers, but in ways quite different from ours. One of the early investigators of the law for space of two and three dimensions was Bravais, whose essay, *Sur les probabilité's des erreurs de situation d' un point*, may be found in the *Mémoires . . . par divers savans . . . .* , Inst. France, Vol. IX. (1846). His process treats the probability of error of an observed point in one direction as independent of its probability of error in another direction perpendicular to the first. This objectionable assumption, which has been made by other writers on the subject so far as I know, is avoided in our present method. (See ANALYST, May 1881, pp. 75 and 79.)

ON THE COMPUTATION OF PROBABLE ERROR.

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IN computing the probable error of the determinations of an observed quantity two forms are in common use, Bessel's and Peters'. If with the ordinary notation we let

$m =$ the number of observations,
$r =$ the probable error of a single observation,
$r_0 =$ the probable error of the final result,
$\rho = 0.4769363$,
$v_1, v_2 \ldots =$ the residual errors of observation, $[vv]$ the sum of their squares and $[v$ their sum without reg'd to sign, then accord'g to Bessel's form,

$$r = \frac{\rho\sqrt{2}}{\sqrt{(m-1)}}\sqrt{[vv]} = \frac{0.6745}{\sqrt{(m-1)}}\sqrt{[vv]},$$

$$r_0 = \frac{\rho\sqrt{2}}{\sqrt{m(m-1)}}\sqrt{[vv]} = \frac{0.6745}{\sqrt{m(m-1)}}\sqrt{[vv]},$$

and according to Peters' form

$$r = \rho\sqrt{\left(\frac{\pi}{m(m-1)}\right)}\cdot[v = \frac{0.8453}{\sqrt{m(m-1)}}\cdot[v,$$

$$r_0 = \rho\sqrt{\left(\frac{\pi}{m^2(m-1)}\right)}\cdot[v = \frac{0.8453}{m\sqrt{(m-1)}}\cdot[v.$$

The latter form can be more rapidly computed and is usually close enough.

The labor of computing the factors $0.6745 \div \sqrt{m(m-1)}$, ... is considerable in any case. Accordingly in practice I have found it convenient to tabulate these quantities for values of m from 2 to 100. The formulæ for the probable error being now written in the simple forms

$$r = \lambda_1\sqrt{[vv]} \qquad r = \lambda'[v$$

$$r_0 = \lambda_2\sqrt{[vv]} \qquad r_0 = \lambda''[v,$$

all that we have to do is to enter the proper table with the argument m, take out the corresponding multiplier λ and perform a single multiplication.

For finding $\sqrt{[vv]}$ a close enough approximation to the square root can be taken out at sight from a table of squares, or if it be preferred the computation may be made logarithmically.

It often happens that we wish to find the probable error of a series of observations roughly and quickly. The following rule will be found convenient in such a case and also as a check on the values found by the preceding method.

A glance at the observed results will show the greatest and least, and their difference will be the *range* of the results. Then for the probable error (r) of a single observation we have, if the number of results is

2,	$r = \frac{1}{2}$	the range,
3,	$r = \frac{1}{3}$	" "
between 3 and 8,	$r = \frac{1}{4}$	" "
" 8 " 15,	$r = \frac{1}{5}$	" "
" 15 " 35,	$r = \frac{1}{6}$	" "
" 35 " 100,	$r = \frac{1}{7}$	" "

The probable error of the final result is found at once by dividing the probable error of a single observation by the square root of the number of observations.

In finding the range caution must be exercised with regard to abnormal results.

TABLE I.

m	λ_1	λ_2	m	λ_1	λ_2
			40	0.1080	0.0171
			41	.1066	.0167
2	0.6745	0.4769	42	.1053	.0163
3	.4769	.2754	43	.1041	.0159
4	.3894	.1947	44	.1029	.0155
5	0.3372	0.1508	45	0.1017	0.0152
6	.3016	.1231	46	.1005	.0148
7	.2754	.1041	47	.0994	.0145
8	.2549	.0901	48	.0984	.0142
9	.2385	.0795	49	.0974	.0139
10	0.2248	0.0711	50	0.0964	0.0136
11	.2133	.0643	51	.0954	.0134
12	.2029	.0587	52	.0944	.0131
13	.1947	.0540	53	.0935	.0128
14	.1871	.0500	54	.0926	.0126
15	0.1803	0.0465	55	0.0918	0.0124
16	.1742	.0435	56	.0909	.0122
17	.1686	.0409	57	.0901	.0119
18	.1636	.0386	58	.0893	.0117
19	.1590	.0365	59	.0886	.0115
20	0.1547	0.0346	60	0.0878	0.0113
21	.1508	.0329	61	.0871	.0111
22	.1472	.0314	62	.0864	.0110
23	.1438	.0300	63	.0857	.0108
24	.1406	.0287	64	.0850	.0106
25	0.1377	0.0275	65	0.0843	0.0105
26	.1349	.0265	66	.0837	.0103
27	.1323	.0255	67	.0830	.0101
28	.1298	.0245	68	.0824	.0100
29	.1275	.0237	69	.0818	.0098
30	0.1252	0.0229	70	0.0812	0.0097
31	.1231	.0221	71	.0806	.0096
32	.1211	.0214	72	.0800	.0094
33	.1192	.0208	73	.0795	.0093
34	.1174	.0201	74	.0789	.0092
35	0.1157	0.0196	75	0.0784	0.0091
36	.1140	.0190	80	.0759	.0085
37	.1124	.0185	85	.0736	.0080
38	.1109	.0180	90	.0713	.0075
39	.1094	.0175	100	.0678	.0068

TABLE II.

m	λ'	λ''	m	λ'	λ''
			40	0.0214	0.0034
			41	.0209	.0033
2	0.5978	0.4227	42	.0204	.0031
3	.3451	.1993	43	.0199	.0030
4	.2440	.1220	44	.0194	.0029
5	0.1890	0.0845	45	0.0190	0.0028
6	.1543	.0630	46	.0186	.0027
7	.1304	.0493	47	.0182	.0027
8	.1130	.0399	48	.0178	.0026
9	.0996	.0332	49	.0174	.0025
10	0.0891	0.0282	50	0.0171	0.0024
11	.0806	.0243	51	.0167	.0023
12	.0736	.0212	52	.0164	.0023
13	.0677	.0188	53	.0161	.0022
14	.0627	.0167	54	.0158	.0022
15	0.0583	0.0151	55	0.0155	0.0021
16	.0546	.0136	56	.0152	.0020
17	.0513	.0124	57	.0150	.0020
18	.0483	.0114	58	.0147	.0019
19	.0457	.0105	59	.0145	.0019
20	0.0434	0.0097	60	0.0142	0.0018
21	.0412	.0090	61	.0140	.0018
22	.0393	.0084	62	.0137	.0017
23	.0376	.0078	63	.0135	.0017
24	.0360	.0073	64	.0133	.0017
25	0.0345	0.0069	65	0.0131	0.0016
26	.0332	.0065	66	.0129	.0016
27	.0319	.0061	67	.0127	.0016
28	.0307	.0058	68	.0125	.0015
29	.0297	.0055	69	.0123	.0015
30	0.0287	0.0052	70	0.0122	0.0015
31	.0277	.0050	71	.0120	.0014
32	.0268	.0047	72	.0118	.0014
33	.0260	.9045	73	.0117	.0014
34	.0252	.0043	74	.0115	.0013
35	0.0245	0.0041	75	0.0113	0.0013
36	.0238	.0040	80	.0106	.0012
37	.0232	.0038	85	.0100	.0011
38	.0225	.0037	90	.0095	.0010
39	.0220	.0035	100	.0085	.0008

Example.—In the telegraphic determination of the longitude between St. Paul and Duluth, Minn., June 15, 1871, the following were the corrections found for chronometer No. 176, at 15h. 51m., sidereal time, from the observation of 21 time stars. (See Annual Report of Survey of N. & N. W. Lakes, 1872, p. 40.)

Correction	v	vv
s		
—8.78	+ 0.04	0.0016
.76	+ .02	4
.85	+ .11	121
.78	+ .04	16
.51	— .23	529
.64	— .10	100
.68	— .06	36
.63	— .11	121
.58	— .16	256
.80	+ .06	36
.75	+ .01	1
.78	+ .04	16
.96	+ .22	484
.64	— .10	100
.65	— .09	81
.83	+ .09	81
.70	— .04	16
.64	+ .10	100
.79	+ .05	25
.90	+ .16	256
— 8.93	+ .19	0.0361
Mean — 8.74	2.02	0.2756

Then

$$m = 21$$
$$[v = 2.02$$
$$[vv] = 0.2756$$
$$\sqrt{[vv]} = 0.525, \text{ from a table of squares.}$$

Hence from table I,

$$r = 0.525 \times 0.151 = 0.079$$
$$r_0 = 0.525 \times 0.033 = 0.017.$$

From table II,

$$r = 2.02 \times 0.041 = 0.082$$
$$r_0 = 2.02 \times 0.009 = 0.018$$

Approximate Method.—Range $= 8.96 - 8.51 = 0.45$

$$\therefore r = \frac{0.45}{6} = 0.075$$

$$r_0 = \frac{0.075}{\sqrt{21}} = 0.016.$$

So close an agreement among the determinations of the values of r and r_0 is not always to be expected.